Problem 1. Let G be a finite group. Show that $f \in L^2(G)$ satisfies $f(g) = 1/|G|$ for all $g \in G$ if and only if the Fourier transform satisfies

$$f^{\mathcal{F}}(X) = 0 \quad \text{for all non-trivial irreducible matrix representations } X \text{ of } G,$$

and

$$f^{\mathcal{F}}(X_{\text{triv}}) = 1 \quad \text{where } X_{\text{triv}} \text{ is the 1-dimensional trivial matrix representation of } G.$$

(Hint: use Fourier inversion to show one direction. To show the other, demonstrate that it is sufficient to prove this for only unitary representations X and then use Schur orthogonality.)

Problem 2. A function $p \in L^2(G)$ is said to be a probability measure on G if $p(g) \in [0,1]$ for all $g \in G$ and $\sum_{g \in G} p(g) = 1$. The function $u \in L^2(G)$ defined by $u(g) = 1/|G|$ for all $g \in G$ is said to be the uniform probability measure on G.

Using the definition of group convolution, prove that for any finite group G, we have $u*u = u$.

(It is not required in order to solve the problem, but this can be interpreted probabilistically: if h and k are random elements of the group chosen according to the uniform probability measure, then this result says that the product hk will be uniformly distributed in the group.)

Problem 3. Show that if p is any probability measure on $\mathbb{Z}(N)$ and $p * p = u$, then we must have $p = u$, where as above u is the uniform probability measure on $\mathbb{Z}(N)$.

(Hint: take the Fourier transform of $p * p$ with respect to the 1-dimensional irreducible representations of $\mathbb{Z}(N)$. Show that $p^{\mathcal{F}}(X) = 0$ for X irreducible unless X is the trivial representation.)

(It is not required in order to solve the problem, but this can be interpreted probabilistically: you will have shown for any probability measure p on $\mathbb{Z}(N)$ which is not uniform to begin with, if a and b are randomly chosen according to p then the sum $a + b$ will also not be uniform.)

Problem 4. Let p be the probability measure on S_3 given by

$$p(\varepsilon) = 1/6,\ p((12)) = 1/6,\ p((13)) = 2/9,\ p((23)) = 1/9,\ p((123)) = 2/9,\ p((132)) = 1/9.$$

a) Compute $p^{\mathcal{F}}(X_1)$, $p^{\mathcal{F}}(X_2)$, and $p^{\mathcal{F}}(X_3)$, where X_1 is the 1-dimensional trivial representation of S_3, X_2 is the 1-dimensional sign representation of S_3, and X_3 is the 2-dimensional standard representation computed in Problem 1b) of Homework 2.

b) Show that $p * p = u$ where u is the uniform probability measure on S_3. *(Hint: use the computation in part a) and Problem 1.)*

c) Say a sentence about why the proof in Problem 3 does not apply for S_3.

(It is not required in order to solve the problem, but this can be interpreted probabilistically: you've found a probability measure p on S_3 which is not uniform, but such that if π and σ are chosen according to p, the composition $\pi\sigma$ will be uniformly distributed. Said another way: there is a random shuffle which is not itself uniform, but if the random shuffle is done twice the result is uniform!)

Problem 5. Let M^λ be the permutation module associated to a partition λ.

a) Show that the characters of M^λ are integer-valued functions for all partitions λ.

b) Show that every irreducible character of S_n is an integer-valued function. *(Hint: consider the modules M^λ in turn, with λ taken in reverse lexicographic order. Use the decomposition of M^λ into a direct sum of Specht modules, with S^λ appearing exactly once in the direct sum.)*

Problem 6. Recall the notation that a partition $\lambda = (\lambda_1, \lambda_2, ..., \lambda_\ell)$ can be written $\lambda = 1^{m_1} 2^{m_2} \cdots n^{m_n}$ if λ has m_1 parts equal to 1, m_2 parts equal to 2, etc. For $\lambda \vdash n$ let ϕ^λ be the character of the permutation module M^λ and let $\pi \in S_n$ have cycle type λ. Show that

$$\phi^\lambda(\pi) = m_1! m_2! \cdots m_n!$$

(Hint: this can be done directly by reasoning combinatorially, but it can also be done using the formula proved in class for the character of an induced representation. You may use in your homework the fact that that the permutation module M^λ corresponds to the trivial representation of the subgroup $S_\lambda = S_{\{1,\ldots,\lambda_1\}} \times \cdots \times S_{\{\lambda_1+\cdots+\lambda_{\ell-1}+1,\ldots,\lambda_1+\cdots+\lambda_\ell\}}$ induced to S_n; see Sec. 2.1 of Sagan.)

Problem 7. a) Compute the partition λ and the Young tableaux (P, Q) obtained by applying the Robinson-Schensted algorithm to the permutation $\sigma = 6\ 5\ 4\ 8\ 3\ 9\ 2\ 1\ 7 \in S_9$ (written in one-line notation).

b) Apply the inverse Robinson-Schensted algorithm to the pair of Young tableaux

$P =$

1	4	7	8
2	6	9	
3			
5			

$Q =$

1	2	3	6
4	5	9	
7			
8			

to find the associated permutation $\sigma \in S_9$. (Write σ in one-line notation.)