1. (20 points total) In one dimension, a particle of mass m is in a harmonic potential with potential energy $V_0(x) = \frac{1}{2}m\omega_0^2 x^2$. For times $t < 0$ the system is in its ground state. Then at $t = 0$, the potential is suddenly changed to $V_1(x) = \frac{1}{2}m\omega_1^2 x^2$. Here ω_0 and ω_1 are constants that represent the classical frequency of oscillation.
(a) (5 points) What is $\langle E\rangle$ right before this change?
(b) (5 points) What is $\langle E\rangle$ right after this change? *Hint: (i) The state of the system does not discontinuously change and so right after the change it has the same wave function it had right before the change. (ii) For all energy eigenstates of a harmonic oscillator,* $\langle V(x)\rangle = \langle K\rangle$, *where* $V(x)$ *is the potential energy and* K *is the kinetic energy (iii) For a harmonic oscillator, relate* $\langle V(x)\rangle$ *and* $\langle K\rangle$ *to* $\langle E\rangle$.
(c) (5 points) What is $\langle E\rangle$ at a function of time t where $t > 0$. *Hint: use the equation for* $\frac{d}{dt}\langle\hat{Q}\rangle$ *from the formula page below to determine how* $\langle E\rangle$ *depends on time.*
(d) (5 points) For any time $t > 0$, what is the probability of measuring the energy to be the new ground state energy $E = \hbar\omega_1/2$?

2. (20 points total) A free particle in one dimension of mass m has a wavefunction at $t = 0$

$$\psi(x) = \frac{1}{(2\pi\sigma^2)^{1/4}}\exp\left(\frac{-x^2}{4\sigma^2}\right)\exp(ikx) \tag{1}$$

Here "free particle" means that the potential energy is zero. k and σ are real constants.
(a) (10 points) Calculate $\langle p\rangle$ at $t = 0$.
(b) (10 points) Calculate $\langle p\rangle$ for any time $t > 0$. *Hint: use the equation for* $\frac{d}{dt}\langle\hat{Q}\rangle$ *from the formula page below to determine how* $\langle p\rangle$ *depends on time.*

3. (20 points total) A particle has two states that it can be in. One is where it is at a site A, and the other at an adjacent site B. These can be represented by the two basis states

$$\psi_A = \begin{pmatrix}1\\0\end{pmatrix} \text{ and } \psi_B = \begin{pmatrix}0\\1\end{pmatrix} \tag{2}$$

meaning that in state ψ_A, the particle is only at site A and in state ψ_B, that the particle is only at site B.
The general state of the system is a linear combination of these two states. In this basis, the Hamiltonian is

$$\hat{H} = \hbar\omega\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} \tag{3}$$

where ω is a constant with units of frequency.
(a) (10 points) Find the energy eigenstates and eigenvalues for this Hamiltonian.
(b) (10 points) Assume that at $t = 0$, the particle is observed to be on site A. Calculate the probability that if observed again, at a time t, that the particle will be found to be on site B.

4. (20 points total) A particle, with mass m is confined to a two dimensional rectangular box of length $2a$ along the x axis, and a along the y axis. That is, the potential energy is 0 if $0 < x < 2a$ and $0 < y < a$, and is infinite otherwise.
(a) (10 points) Calculate the ground state and first excited state wave functions and their corresponding energies. For this part of the problem, you do not need to consider the particle's spin.
(b) (10 points) Consider the same situation but with the addition of a second identical particle. Both particles are spin 1/2 fermions. Calculate the total ground state energy and two particle wave function. This wave function should include the spin.

5. (20 points)
Two spin 1/2 particles are prepared in the state $|\rightarrow\rightarrow\rangle$ that is, each spin is in an eigenstate of its S_x operator with eigenvalue $\hbar/2$. This is analogous to the notation $|\uparrow\uparrow\rangle$ representing each spin being in an eigenstate of its S_z operator with eigenvalue $\hbar/2$.
What is the probability that the system will be observed in the state $|sm\rangle$ where s is the total angular momentum and m is the total angular momentum in the z direction. Find the probabilities for all four possible states.

1. (20 points total) In one dimension, a particle of mass m is in a harmonic potential with potential energy $V_0(x) = \frac{1}{2}m\omega_0^2 x^2$. For times $t < 0$ the system is in its ground state. Then at $t = 0$, the potential is suddenly changed to $V_1(x) = \frac{1}{2}m\omega_1^2 x^2$. Here ω_0 and ω_1 are constants that represent the classical frequency of oscillation.

(a) (5 points) What is $\langle E \rangle$ right before this change?

(b) (5 points) What is $\langle E \rangle$ right after this change? *Hint: (i) The state of the system does not discontinuously change and so right after the change it has the same wave function it had right before the change. (ii) For all energy eigenstates of a harmonic oscillator, $\langle V(x) \rangle = \langle K \rangle$, where $V(x)$ is the potential energy and K is the kinetic energy (iii) For a harmonic oscillator, relate $\langle V(x) \rangle$ and $\langle K \rangle$ to $\langle E \rangle$.*

(c) (5 points) What is $\langle E \rangle$ at a function of time t where $t > 0$. *Hint: use the equation for $\frac{d}{dt}\langle \hat{Q} \rangle$ from the formula page below to determine how $\langle E \rangle$ depends on time.*

(d) (5 points) For any time $t > 0$, what is the probability of measuring the energy to be the new ground state energy $E = \hbar\omega_1/2$?

(a) For the system is in ground state, we have, for HO,

$$\hat{H}_0|\psi_0\rangle = E_0|\psi_0\rangle = \frac{1}{2}\hbar\omega_0|\psi_0\rangle \quad , H_0 = \frac{p^2}{2m} + \frac{1}{2}m\omega_0^2x^2$$

$$\therefore \langle E \rangle = E_0 = \frac{1}{2}\hbar\omega_0$$

(b) right after change, $|\psi'\rangle = |\psi_0\rangle$, and,

$$\hat{H}'|\psi_0\rangle = (K + \frac{1}{2}m\omega_1^2x_1^2)|\psi_0\rangle$$

For $\langle H' \rangle = \langle\psi_0|K + \frac{1}{2}m\omega_1^2x_1^2|\psi_0\rangle$

$$= 2\langle\psi_0|\frac{1}{2}m\omega_1^2x_1^2|\psi_0\rangle = \langle E' \rangle$$

For before the change,

$$\hat{H}_0|\psi_0\rangle = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}|\psi_0\rangle + \frac{1}{2}m\omega_0^2x^2|\psi_0\rangle = E_0|\psi_0\rangle$$

$$\therefore |\psi_0\rangle = \left(\frac{m\omega_0}{\pi\hbar}\right)^{\frac{1}{4}} \exp\left(-\frac{m\omega_0}{2\hbar}x^2\right)$$, satisfy this with $E_0 = \frac{1}{2}\hbar\omega_0$

so after the change

$$\langle E' \rangle = 2\langle\psi_0|\frac{1}{2}m\omega_1^2x^2|\psi_0\rangle$$

$$= 2\int_{-\infty}^{\infty} \psi_0^* \cdot \frac{1}{2}m\omega_1^2x^2 \cdot \psi_0 \, dx$$

$$= 2\int_{-\infty}^{\infty} \left(\frac{m\omega_0}{\pi\hbar}\right)^{\frac{1}{2}} \frac{1}{2}m\omega_1^2x^2 \cdot \exp\left(-\frac{m\omega_0}{\hbar}x^2\right) dx$$

$$= 2\left(\frac{m\omega_0}{\pi\hbar}\right)^{\frac{1}{2}} \frac{1}{2}m\omega_1^2 \cdot \int_{-\infty}^{+\infty} x^2 \exp\left(-\frac{m\omega_0}{\hbar}x^2\right) dx$$

$$= 2\left(\frac{m\omega_0}{\pi\hbar}\right)^{\frac{1}{2}} \frac{1}{2} m\omega_1^2 \cdot \left(\frac{\sqrt{\pi}\,\mathrm{erf}(\sqrt{a}\,x)}{4a^{3/2}} - \frac{x}{2a} e^{-ax^2}\right)\Big|_{-\infty}^{\infty}$$

$$= 2\left(\frac{m\omega_0}{\pi\hbar}\right)^{\frac{1}{2}} \frac{1}{2} m\omega_1^2 \frac{\sqrt{\pi}}{2\left(\frac{m\omega_0}{\hbar}\right)^{\frac{3}{2}}}$$

$$= \frac{\hbar\,\omega_1^2}{2\,\omega_0\sqrt{\pi}}$$

(c) $\frac{d\langle E\rangle}{dt} = \frac{i}{\hbar}\langle [\hat{H}, E]\rangle + \langle \frac{\partial E}{\partial t}\rangle$

$\langle E\rangle = \int_{-\infty}^{\infty} \psi^*(x,t)\,\hat{H}\,\Psi(x,t)$

For $\psi(x, t=0) = \psi_0 = \left(\frac{m\omega_0}{\pi\hbar}\right)^{\frac{1}{4}} \exp\left(-\frac{m\omega_0}{2\hbar}x^2\right)$,

so, we have, given $\hat{H}\psi = i\hbar\frac{\partial}{\partial t}\psi$,

$$\psi(x,t) = e^{-\frac{iH}{\hbar}t}\,\psi(x,0)$$

For For $H = \frac{p^2}{2m} + \frac{1}{2}m\omega_1^2 x^2$, set $H|\phi_i\rangle = E_i|\phi_i\rangle$

$\therefore \psi(x,0) = \sum_n c_n |\phi_i\rangle$

we have, $\psi(x,t) = \sum_n c_n e^{-\frac{iH}{\hbar}t}|\phi_i\rangle$

$$= \sum_n c_n e^{-\frac{iE_i t}{\hbar}}|\phi_i\rangle$$

$\therefore \langle E\rangle = \langle \psi^*(x,t)|H|\Psi(x,t)\rangle$

$$= \left(\sum_n c_n e^{-\frac{iE_i t}{\hbar}}|\phi_i\rangle\right)^* \hat{H}\left(\sum_n c_n e^{-\frac{iE_i t}{\hbar}}|\phi_i\rangle\right)$$

dx

$$= \sum_n C_n^2 \tilde{E}_n = \text{const}, \text{ so},$$

$$\langle E\rangle = \langle E\rangle\big|_{t=0} = \langle \psi_0 | H | \psi_0\rangle = \frac{\hbar w_1^2}{2 w_0 \sqrt{\pi}},$$

does not depend on time

also, $\frac{d\langle E\rangle}{dt} = \frac{i}{\hbar}\langle [H, E]\rangle + \langle \frac{\partial E}{\partial t}\rangle$

$$= \frac{i}{\hbar}\langle [H, H]\rangle + \langle \frac{\partial H}{\partial t}\rangle, \quad \because H = \text{const}, \text{ so},$$

$$\frac{\partial H}{\partial t} = 0, \quad [H, H] = 0, \text{ so}, \quad \frac{dE}{dt} = 0$$

$$E = \text{const} = E_{initial} = \frac{\hbar w_1^2}{2 w_0 \sqrt{\pi}}$$

(d)

$\psi(x,0) = \sum_n C_n \phi_i(x,0)$, $\phi_i(x,0)$ is solution of $H|\phi_i\rangle = \tilde{E}_i|\phi_i\rangle$

where, $H = \frac{p^2}{2m} + \frac{1}{2} m w_1^2 x^2$, we can derive,

$|\phi_1\rangle = \left(\frac{m w_1}{\pi\hbar}\right)^{\frac{1}{4}} \exp\left(-\frac{m w_1}{2\hbar} x^2\right)$

For

$$\psi(x,t)\rangle = e^{-\frac{iHt}{\hbar}} \psi(x,0) = e^{-\frac{iHt}{\hbar}} \sum_n C_n \phi_i(x,0)$$

$$= \sum_n C_n e^{-\frac{iH}{\hbar}t} \phi_i(x,0)$$

$$= \sum_n C_n e^{-\frac{i\tilde{E}_i t}{\hbar}} \phi_i(x,0)$$

$$\therefore H\psi(x,t) = \sum_n E_1 C_n e^{-\frac{i\tilde{E}_i t}{\hbar}} \phi_i(x,0)$$

so, the probability to measure $H = \tilde{E}_1$ is:

$$P(H = E_1) = C_1^2 = \langle \phi_1 | \psi(x,t)\rangle^2$$

$$= \langle \phi_1 | \psi(x,0)\rangle^2$$

$$= \langle \phi_1 | \psi_0 \rangle^2$$

$$= \left(\int_{-\infty}^{\infty} \left(\frac{m\omega_1}{\pi\hbar}\right)^{\frac{1}{4}} \exp\left(-\frac{m\omega_1}{2\hbar}x^2\right) \left(\frac{m\omega_0}{\pi\hbar}\right)^{\frac{1}{4}} \exp\left(-\frac{m\omega_0}{2\hbar}x^2\right) \right)^2$$

$$= \left(\frac{m}{\pi\hbar}\right) (\omega_1\omega_0)^{\frac{1}{2}} \cdot \left(\frac{\sqrt{\pi}}{\sqrt{\frac{m}{2\hbar}(\omega_1+\omega_0)}} \right)^2$$

$$= \left(\frac{2}{\pi}\right) (\omega_1\omega_0)^{\frac{1}{2}} \frac{\pi}{(\omega_1+\omega_0)}$$

$$= \frac{2\sqrt{\omega_1\omega_0}}{\omega_1+\omega_0}$$

2. (20 points total) A free particle in one dimension of mass m has a wavefunction at $t=0$

$$\psi(x) = \frac{1}{(2\pi\sigma^2)^{1/4}} \exp\left(\frac{-x^2}{4\sigma^2}\right) \exp(ikx) \tag{1}$$

Here "free particle" means that the potential energy is zero. k and σ are real constants.
(a) (10 points) Calculate $\langle p \rangle$ at $t = 0$.
(b) (10 points) Calculate $\langle p \rangle$ for any time $t > 0$. *Hint: use the equation for $\frac{d}{dt}\langle \hat{Q} \rangle$ from the formula page below to determine how $\langle p \rangle$ depends on time.*

(a) $$\langle p \rangle = \int_{-\infty}^{\infty} \psi^* \left(-i\hbar \frac{\partial}{\partial x}\right) \psi \, dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{(2\pi\sigma^2)^{\frac{1}{2}}} \exp\left(\frac{-x^2}{4\sigma^2}\right) \exp(-ikx) \left(-i\hbar \frac{\partial}{\partial x}\right) \left(\exp\left(\frac{-x^2}{4\sigma^2}\right) \exp(ikx) \right)$$

$$= -i\hbar \int_{-\infty}^{\infty} \frac{1}{(2\pi\sigma^2)^{\frac{1}{2}}} \exp\left(\frac{-x^2}{4\sigma^2}\right) \exp(-ikx) \cdot \exp\left(-\frac{x^2}{4\sigma^2} + ikx\right) \left(-\frac{x}{2a} + ik\right)$$

$$= -i\hbar \int_{-\infty}^{\infty} \frac{1}{(2\pi\sigma^2)^{\frac{1}{2}}} \exp\left(\frac{-x^2}{2\sigma^2}\right) \left(-\frac{x}{2a} + ik\right) dx$$

$$= i\hbar \int_{-\infty}^{\infty} \frac{1}{(2\pi\sigma^2)^{\frac{1}{2}}} \exp\left(\frac{-x^2}{2\sigma^2}\right) \cdot \frac{x}{2a} dx + \hbar \int_{-\infty}^{\infty} \frac{1}{(2\pi\sigma^2)^{\frac{1}{2}}} \exp\left(\frac{-x^2}{2\sigma^2}\right) k \, dx$$

$$= I_1 + I_2$$

For $I_1 = i\hbar \int_{-\infty}^{\infty} \frac{1}{(2\pi\sigma^2)^{\frac{1}{2}}} \exp\left(\frac{-x^2}{2\sigma^2}\right) \cdot \frac{x}{2a} dx$, and,

$\frac{1}{(2\pi\sigma^2)^{\frac{1}{2}}} \exp\left(\frac{-x^2}{2\sigma^2}\right) \cdot \frac{x}{2a}$ is odd function, so,

$I_1 = i\hbar \int_{-\infty}^{\infty} \frac{1}{(2\pi\sigma^2)^{\frac{1}{2}}} \exp\left(\frac{-x^2}{2\sigma^2}\right) \cdot \frac{x}{2a} dx = 0$, so,

$$\langle p\rangle = \int_{-\infty}^{\infty} \psi^*\left(-i\hbar\frac{\partial}{\partial x}\right)\psi\, dx = \hbar\int_{-\infty}^{\infty}\frac{1}{(2\pi\sigma^2)^{\frac{1}{2}}}\exp\left(\frac{-x^2}{2\sigma^2}\right)k\, dx$$

$$= \hbar k\frac{1}{(2\pi\sigma^2)^{\frac{1}{2}}}\int_{-\infty}^{\infty}\exp\left(\frac{-x^2}{2\sigma^2}\right)dx$$

$$= \hbar k\frac{1}{\sqrt{2\pi\sigma^2}}\cdot\int\exp\left(-\frac{x^2}{2\sigma^2}\right)d\frac{x}{\sqrt{2\sigma^2}}\cdot\sqrt{2\sigma^2}$$

$$= \hbar k\frac{\sqrt{2\pi\sigma^2}}{\sqrt{2\pi\sigma^2}} = \hbar k$$

$$\therefore \langle p\rangle = \hbar k$$

(b) For,

$$\frac{d\langle p\rangle}{dt} = \frac{i}{\hbar}\langle[H,p]\rangle + \left\langle\frac{\partial p}{\partial t}\right\rangle, \quad H = \frac{p^2}{2m}$$

$\because [H,p] = \left[\frac{p^2}{2m}, p\right] = 0$ For free particle,

$\because \left\langle\frac{\partial p}{\partial t}\right\rangle = \frac{\partial(\hbar k)}{\partial t} = 0$, so,

$\langle p\rangle = \langle p\rangle_{\text{initial}} = \hbar k$, $\langle p\rangle$ is conserved, so,

we have for any time t,

$$\langle p\rangle = \hbar k$$

3. (20 points total) A particle has two states that it can be in. One is where it is at a site A, and the other at an adjacent site B. These can be represented by the two basis states

$$\psi_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \psi_B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2}$$

meaning that in state ψ_A, the particle is only at site A and in state ψ_B, that the particle is only at site B. The general state of the system is a linear combination of these two states. In this basis, the Hamiltonian is

$$\hat{H} = \hbar\omega \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3}$$

where ω is a constant with units of frequency.

(a) (10 points) Find the energy eigenstates and eigenvalues for this Hamiltonian.

(b) (10 points) Assume that at $t = 0$, the particle is observed to be on site A. Calculate the probability that if observed again, at a time t, that the particle will be found to be on site B.

(a) For $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\det\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \lambda E\right)$

$= \det\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - 1 = 0$, λ is eigenvalue,

so, $\lambda_1 = 1$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 1 \cdot \begin{pmatrix} a \\ b \end{pmatrix}$, $|\psi_1\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$

we get eigenstate, $|\psi_1\rangle = \begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(\psi_A + \psi_B)$

for $\lambda = -1$, we get,

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a' \\ b' \end{pmatrix} = -1 \begin{pmatrix} a' \\ b' \end{pmatrix}$, with $|\psi_2\rangle = \begin{pmatrix} a' \\ b' \end{pmatrix}$

$\therefore \begin{pmatrix} a' \\ b' \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = |\psi_2\rangle = \frac{1}{\sqrt{2}}(\psi_A - \psi_B)$

For $H = \hbar\omega\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, we get,

	eigen value	eigen state
①	$\hbar\omega$	$\frac{1}{\sqrt{2}}(\psi_A + \psi_B) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
②	$-\hbar\omega$	$\frac{1}{\sqrt{2}}(\psi_A - \psi_B) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(b) For TISE: $i\hbar\frac{\partial|\psi\rangle}{\partial t} = \hat{H}|\psi\rangle$, so, we have,

$$|\psi(t)\rangle = e^{-\frac{iHt}{\hbar}} |\psi(0)\rangle$$

For $t=0$, set $|\psi_1\rangle = \frac{1}{\sqrt{2}}(\psi_A + \psi_B)$, $|\psi_2\rangle = \frac{1}{\sqrt{2}}(\psi_A - \psi_B)$

are eigenfunction of H,

$$\psi(0) = \psi_A = \binom{1}{0} = \left(\frac{1}{\sqrt{2}}(\psi_A + \psi_B) + \frac{1}{\sqrt{2}}(\psi_A - \psi_B)\right)\frac{1}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}}(\psi_1 + \psi_2)$$

$$\therefore \quad \psi(t) = e^{-\frac{iHt}{\hbar}}\left(\frac{1}{\sqrt{2}}(\psi_1 + \psi_2)\right)$$

$$= \frac{1}{\sqrt{2}} e^{-\frac{iHt}{\hbar}} \psi_1 + \frac{1}{\sqrt{2}} e^{-\frac{iHt}{\hbar}} \psi_2$$

$$= \frac{1}{\sqrt{2}} e^{-\frac{i\hbar\omega t}{\hbar}} \binom{1}{1} + \frac{1}{\sqrt{2}} e^{\frac{i\hbar\omega t}{\hbar}} \binom{1}{-1}$$

$\therefore$ probability at time t to be on site B is: $\langle\psi_B| = (0,1)$

$$p(B) = |\langle\psi_B|\psi(t)\rangle|^2$$

$$= \left(\frac{1}{\sqrt{2}} e^{-\frac{i\hbar\omega t}{\hbar}} (0\ \ 1)\cdot\binom{1}{1} + \frac{1}{\sqrt{2}} e^{\frac{i\hbar\omega t}{\hbar}} (0,1)\cdot\binom{1}{-1}\right)^2$$

$$= \left(\frac{1}{\sqrt{2}} e^{-\frac{i\hbar\omega t}{\hbar}} - \frac{1}{\sqrt{2}} e^{\frac{i\hbar\omega t}{\hbar}}\right)$$

$$= \left(\frac{1}{\sqrt{2}}\cdot 2\sin\omega t\right)^2$$

$$= 2\sin^2\omega t$$

4. (20 points total) A particle, with mass m is confined to a two dimensional rectangular box of length $2a$ along the x axis, and a along the y axis. That is, the potential energy is 0 if $0 < x < 2a$ and $0 < y < a$, and is infinite otherwise.

(a) (10 points) Calculate the ground state and first excited state wave functions and their corresponding energies. For this part of the problem, you do not need to consider the particle's spin.

(b) (10 points) Consider the same situation but with the addition of a second identical particle. Both particles are spin 1/2 fermions. Calculate the total ground state energy and two particle wave function. This wave function should include the spin.

(a) For static state, TISE is:

$$-\frac{\hbar^2}{2m}\nabla^2\psi(x,y) = E\psi(x,y), \text{ at } \begin{cases} 0<x<2a \\ 0<y<a \end{cases}$$

and, For else, $V\to\infty$, we suppose, $\psi(x,y)\to 0$, so,

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi(x,y)}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) = E\psi(x,y)$$

set, $\psi(x,y) = X(x)\,Y(y)$, so, we get,

$$-\frac{\hbar^2}{2m}\frac{d^2X(x)}{dx^2} = \varepsilon_x X(x),$$

$$-\frac{\hbar^2}{2m}\frac{d^2Y(y)}{dy^2} = \varepsilon_y Y(y),$$

$E = \varepsilon_x + \varepsilon_y$, and we get general solution,

$$k_x = \sqrt{\frac{2mE_x}{\hbar^2}}, \quad k_y = \sqrt{\frac{2mE_y}{\hbar^2}}$$

$$X(x) = A_x \sin k_x x + B_x \cos k_x x$$

$$Y(y) = A_y \sin k_y y + B_y \cos k_y y$$

For $\psi(0, y) = \psi(x, 0) = \psi(2a, y)$
$= \psi(x, a) = 0$, we get,

$X(0) = X(2a) = 0$,
$Y(0) = Y(2a) = 0$, so, we get,

$B_x = B_y = 0$, and,

$$\sin\left(\sqrt{\frac{2mE_x}{\hbar^2}} \cdot 2a\right) = \sin\left(\sqrt{\frac{2mE_y}{\hbar^2}} \cdot a\right) = 0$$

$$\therefore E_{nx} = \frac{\hbar^2 \pi^2}{8ma^2} n_x^2,$$

$$E_{ny} = \frac{\hbar^2 \pi^2}{2ma^2} n_y^2, \quad n_x, n_y = 1, 2 \ldots$$

we get, $\psi(x,y) = N \cdot \sin\left(\frac{n_x \pi x}{2a}\right) \sin\left(\frac{n_y \pi y}{a}\right)$

$\because \iint |\psi(x,y)|^2 \, ds = 1,$

$\therefore N^2 \int_0^{2a} \sin^2\left(\frac{n_x \pi x}{2a}\right) dx \int_0^{a} \sin\left(\frac{n_y \pi y}{a}\right) dy = 1$

so, $N^2 \cdot \frac{2a}{2} \cdot \frac{a}{2} = 1$

$\therefore N = \frac{\sqrt{2}}{a}$

$\therefore \psi_{n_x, n_y} = \frac{\sqrt{2}}{a} \cdot \sin\left(\frac{n_x \pi x}{2a}\right) \sin\left(\frac{n_y \pi y}{a}\right)$

the ground state, $n_x = n_y = 1$, so

$$E_{11} = \varepsilon_{1x} + \varepsilon_{1y} = \frac{5\hbar^2\pi^2}{8ma^2},$$

$$\psi_{11} = \frac{\sqrt{2}}{a} \sin\left(\frac{\pi x}{2a}\right) \sin\left(\frac{\pi y}{a}\right)$$

First excited state: $n_x=2,\ n_y=1$

$$E_{21}=\frac{4\hbar^2\pi^2}{8ma^2}+\frac{4\hbar^2\pi^2}{8ma^2}n_x^2$$

$$=\frac{\hbar^2\pi^2}{ma^2}$$

$$\therefore\ \psi_{n_x=2,n_y=1}=\frac{\sqrt{2}}{a}\sin\left(\frac{2\pi x}{2a}\right)\sin\left(\frac{\pi y}{a}\right)$$

(b) For spin state $|S_\uparrow\rangle=\frac{\hbar}{2}\begin{pmatrix}1\\0\end{pmatrix}$ or $|S_\downarrow\rangle=\frac{\hbar}{2}\begin{pmatrix}0\\1\end{pmatrix}$,

up and down, we get, total wave function is

$|n_x,n_y,s\rangle=\Phi=|\psi\rangle\otimes|s\rangle$, For particle 1 combined

with particle 2, we have,

$$|n_{x1},n_{y1},s_1,n_{x2},n_{y2},s_2\rangle=(|\psi_1\rangle\otimes|s_1\rangle)\otimes(|\psi_2\rangle\otimes s_2)$$

and, For two Fermion,

$$|n_{x2},n_{y2},s_2\ n_{x1},n_{y1},s_1\rangle=-|n_{x1},n_{y1},s_1\ n_{x2},n_{y2},s_2\rangle,$$

we get,

$$\psi(r_1, r_2) = \frac{1}{\sqrt{2}}\left(\psi_{11}(r_1)|s_1\rangle - \psi_{11}(r_2)|s_2\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left[\frac{\sqrt{2}}{a}\sin\left(\frac{\pi x_1}{2a}\right)\sin\left(\frac{\pi y_1}{a}\right)\cdot\begin{pmatrix}\frac{\hbar}{2}\\0\end{pmatrix} - \frac{\sqrt{2}}{a}\sin\left(\frac{\pi x_2}{2a}\right)\sin\left(\frac{\pi y_2}{a}\right)\cdot\begin{pmatrix}0\\\frac{\hbar}{2}\end{pmatrix}\right]$$

5. (20 points)
Two spin 1/2 particles are prepared in the state $|\rightarrow\rightarrow\rangle$ that is, each spin is in an eigenstate of its S_x operator with eigenvalue $\hbar/2$. This is analogous to the notation $|\uparrow\uparrow\rangle$ representing each spin being in an eigenstate of its S_z operator with eigenvalue $\hbar/2$.
What is the probability that the system will be observed in the state $|sm\rangle$ where s is the total angular momentum and m is the total angular momentum in the z direction. Find the probabilities for all four possible states.

For state $|\rightarrow\rightarrow\rangle$, we have,

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}, \text{ so, } |\rightarrow\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

so, we have $|\rightarrow,\rightarrow\rangle$ is

$$|\rightarrow,\rightarrow\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\otimes\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

$$= \left(\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix}\right)\otimes\left(\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix}\right)$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix}\otimes\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix}\otimes\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix}\otimes\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix}\otimes\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix}$$

we get, For $s = \frac{1}{2} + \frac{1}{2} = 1$, so, $m = 0, 1, -1$

For $m = -1$, we have state $\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix}\otimes\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix} = |1,-1\rangle$, with $p(m=-1) = (\frac{1}{2})^2 = \frac{1}{4}$,

For $m=0$, we have, $|1,0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix}\otimes\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix}\otimes\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix}$

$p(m=0) = (\frac{\sqrt{2}}{2})^2 = \frac{1}{2}$, For $m=1$, we have $|1,1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix}$,

with $P(m=1) = (\frac{1}{2})^2 = \frac{1}{4}$

$\therefore$

$\|s\ m\rangle$	probability
$\|1\ -1\rangle$	$\frac{1}{4}$
$\|1\ 0\rangle$	$\frac{1}{2}$
$\|1\ 1\rangle$	$\frac{1}{4}$