MA530a (FALL 2021)

HOMEWORK #3

Due Sunday, October 24, 2021

1) Using Jensen's Inequality to argue that

   i) $|E[X]| \le E[|X|]$;

   ii) $|E[X]|^2 \le E[|X|^2]$,

   provided that all the expectations involved exist. (A challenge: if you claim that a function is "convex", you must prove it!)

2) Assume that a process $\{X_n : n = 1, 2, \cdots\}$ satisfies the following: (i) $\sup_n E[|X_n|] < \infty$, and (ii) $\forall \varepsilon > 0$, $\exists \delta > 0$, such that

$$E\{|X_n| : A\} < \varepsilon, \qquad \text{whenever } P(A) < \delta.$$

   Show that $\lim_{K\to\infty} \sup_n E\{|X_n|\mathbf{1}_{\{|X_n|>K\}}\} = 0$. Namely $\{X_n\}$ is *uniformly integrable.*

3) Show that if $\{X_n\}$ is uniformly integrable, then

   (i) $\sup_n E([X_n]^+) < \infty$;

   (ii) If we assume further that $\{X_n\}$ is a martingale, then $\lim_n X_n = X_\infty$ exists (by upcrossing theorem), and $E|X_\infty| < \infty$ (Hint: use Fatou).

4) Assume that Z is a random variable with $E|Z| < \infty$ and $\{\mathcal{G}_n\}$ is a filtration. Show that $M_n := E\{Z|\mathcal{G}_n\}$ is an U.I. $\{\mathcal{G}_n\}$-martingale.

5) Let $\{B_t : t \ge 0\}$ be a standard Brownian motion. For any $t \ge 0$ and $n \ge 0$, define

$$Z_n = \sum_{i=1}^{2^n} \Big(B_{it/2^n} - B_{(i-1)t/2^n}\Big)^2 - t.$$

   Calculate EZ_n and $E[Z_n]^2$.

6) §4.9 of Björk's book: all exercises except for #4.6 and 4.7.

7) Let $M_t = \int_0^t h(s)dB_s$, $N_t = \int_0^t g(s)dB_s$, $t \ge 0$, where $h, g \in L^2_{\mathbf{F}}([0,T]\times\Omega)$. Define $\langle M,N\rangle_t := \int_0^t h(s)g(s)ds$, $t \ge 0$. Show that $M_tN_t - \langle M,N\rangle_t$ is a martingale. (Hint: Recall $\langle M\rangle_t = \int_0^t |h(s)|^2 ds$.)

8) Assume that X is a solution to the following SDE:

$$dX_t = b(t,X_t)dt + \sigma(t,X_t)dB_t, \qquad X_0 = x, \quad t \ge 0,$$

   and $u = u(t,x)$ is a classical ($C^{1,2}$) solution to the PDE:

$$\begin{cases} 0 = u_t + \dfrac{1}{2}\sigma^2(t,x)u_{xx} + b(t,x)u_x + c(t)u + f(t,x); \\ u(T,x) = g(x). \end{cases}$$

   Argue that $u(t,x) = E\Big\{e^{\int_t^T c(s)ds} g(X_T) + \int_t^T e^{\int_t^s c(r)dr} f(s,X_s)ds \Big| X_t = x\Big\}$.