1. ① reflexive:

$aRa \Leftrightarrow a^2$ is a perfect square. True.

② symmetric:

$aRb \Leftrightarrow ab$ is a perfect square.

$\Leftrightarrow ba$ is a perfect square. $\Leftrightarrow bRa$

③ transitive:

if aRb & bRc, then

ab, bc are perfect squares

$\Rightarrow ab^2c = b^2(ac)$ are perfect squares,

say $ab^2c = k^2$ for $k \in \mathbb{N}$.

Thus $b^2 | k^2$, i.e. $ac = k^2/b^2 \in \mathbb{N}$

Note that $k/b \in \mathbb{N} \Rightarrow ac$ is a perfect square

$\Rightarrow aRc$.

By ①②③, R is an equivalent relation.

2. For $(x,y) \in \mathbb{Z} \times \mathbb{Z}$, define that

$\mathcal{R} = \{(0,1), (1,0), (1,1), (0,0), (2,2)\}$

$\mathcal{S} = \{(0,1), (1,0), (1,1), (0,0), (3,3)\}$.

It can be checked that

R, S are transitive and symmetric.

And $|\mathcal{R} - \mathcal{S}| = |\{(2,2)\}| = |\mathcal{S} - \mathcal{R}| = |\{(3,3)\}| = 1$.

3. ① Prove f is surjective.

$\forall Q \in B$, say $Q(x) = a_{2020}x^{2020} + a_{2019}x^{2019} + \dots + a_1 x + a_0$.

Then note that by taking

$$P_b(x) = \frac{a_{2020}}{2021}x^{2021} + \frac{a_{2019}}{2020}x^{2020} + \cdots + \frac{a_1}{2}x^2 + a_0x + b$$

where $b \in \mathbb{R}$ a constant, we have $P_b' = Q$.

If $P_b(2) = 3$, then $b = 3 - \sum_{i=0}^{2020} \frac{a_i}{i+1} \cdot 2^{i+1}$, which is determined.

$\Rightarrow \exists P = P_b \in A$ s.t. $f(P) = P_b' = Q$.

$\Rightarrow f$ is surjective.

② Prove that f is injective.

Suppose $f(S) = f(T) = Q \in B$ for $S, T \in A$.

Then $S = P_{b_1}$, $T = P_{b_2}$ with $b_1, b_2 \in \mathbb{R}$ by ①.

However, if $b_1 \neq b_2$, then $P_{b_1} \in A \Rightarrow P_{b_2} \notin A$

& $P_{b_2} \in A \Rightarrow P_{b_1} \notin A$

since b is uniquely determined by $a_0, \cdots, a_{2020}$.

$\Rightarrow b_1 = b_2 \Rightarrow S = T$

$\Rightarrow f$ is injective.

Thus, by ①②, f is a bijection.

4. If f is injective then we have $|B| \geq |A|$.

Again, if h is surjective then we obtain $|B \times B| \leq |A|$

Thus,

$$\left.\begin{array}{l} |B \times B| \leq |A| \leq |B| \\ \text{with } |B \times B| \geq |B| \end{array}\right\} \Rightarrow |B \times B| = |A| = |B|$$

① If $|B| < \infty$, then $|A| \geq 2$ leads to $|B \times B| = |B|^2 = |B| \geq 2$

which is absurd. So $|A| = 1$ in this case.

② If $|B| = \infty$, then $|A| = \infty$ and there is a bijection $\varphi : A \longrightarrow B$.

E.g. $A = B = \mathbb{N}$ is correct.

(So this problem requires A, B to be finite).

5. ① Prove f is injective.

Suppose $f(\frac{a_1}{b_1}) = f(\frac{a_2}{b_2}) \Leftrightarrow \frac{a_1(b_1+1)}{b_1} = \frac{a_2(b_2+1)}{b_2}$

with $\gcd(a_1, b_1) = \gcd(a_2, b_2) = 1$.

On the other hand, may assume $b_1 \neq \pm 1$, $b_2 \neq \pm 1$.

note that $\gcd(b_1, b_1+1) = \gcd(b_2, b_2+1)$,

so it forces $b_1 = b_2$.

Thus, $a_1 = a_2$ as well.

$\Rightarrow a_1/b_1 = a_2/b_2$, f is injective.

② Prove f is not surjective.

Assume it is. Then $\forall\, p/q \in \mathbb{Q}$ with $\gcd(p, q) = 1$,

$\exists\, a/b \in \mathbb{Q}$ s.t. $f(a/b) = a(b+1)/b = p/q$.

Similarly, note that when $b \neq 1$, $q \neq 1$,

we obtain $\gcd(a, b) = \gcd(b+1, b) = 1$

$\Rightarrow \quad b = q \quad \Rightarrow \quad a = \frac{pb}{q \cdot (b+1)} = \frac{p}{q+1}$.

RHS above is not necessarily an integer

since p, q are arbitrary. Contradiction!

Thus, f is not an surjection.

6. Note that if $\exists\, k, l \in \mathbb{N}$ s.t.

$a_k \equiv a_{k+l} \bmod m$ & $a_{k+1} \equiv a_{k+l+1} \bmod m$

then $a_n \equiv a_{n+l} \mod m$ for all $n \geq k$.

This is because in the sense of mod m:

$$a_{k+2} = r a_{k+1} + s a_k \equiv r a_{k+l+1} + s a_{k+l} = a_{k+l+2}$$

$$a_{k+3} = r a_{k+2} + s a_{k+1} \equiv r a_{k+l+2} + s a_{k+l+1} = a_{k+l+3}$$

$$\cdots \quad \cdots$$

$$a_{k+(n-k)} \equiv a_{k+l+(n-k)} \iff a_n \equiv a_{n+l}.$$

Now, consider the map

$$\varphi : \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z} \times \mathbb{Z}$$

$$(a_i, a_{i+1}) \longmapsto (\overline{a_i}, \overline{a_{i+1}})$$

where $\overline{a_i} = (a_i \mod m)$, $\overline{a_{i+1}} = (a_{i+1} \mod m)$

& $0 \leq \overline{a_i}, \overline{a_{i+1}} \leq m$.

It is clear that there are many finitely many pairs of the form $(\overline{a_i}, \overline{a_{i+1}})$,

thus $|\{(\overline{a_i}, \overline{a_{i+1}}) : i \in \mathbb{N}\}| < |\{(a_i, a_{i+1}) : i \in \mathbb{N}\}|$.

By Pigeonhole Principle, $\exists\, i \neq j \in \mathbb{N}$ s.t. $\varphi((a_i, a_{i+1})) = \varphi((a_j, a_{j+1}))$

i.e. $\overline{a_i} = \overline{a_j}$, $\overline{a_{i+1}} = \overline{a_{j+1}}$

i.e. $a_i \equiv a_j \mod m$. $a_{i+1} \equiv a_{j+1} \mod m$.

$\rightsquigarrow$ We can take $|i-j| = l$, $i+1 = k$.

Then we have find $l, k \in \mathbb{N}$ as desired.