A3.

(a) Define the indicator function

$$f(j) = \begin{cases} 1 & \text{if } j \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$$

$$g(j) = \begin{cases} 0 & \text{if } j \text{ is odd} \\ 1 & \text{otherwise} \end{cases}$$

then $g(j) + f(j) = 1 \quad \forall\ 1 \leq j \leq N$

the Langrangian $\mathcal{L}$ is

$$\mathcal{L}(p_0, \cdots, p_8, \lambda_0, \lambda_1, \lambda_2)$$

$$= \sum_{i=0}^{8} p_i \ln p_i - \lambda_0 \cdot \sum_{i=0}^{8} p_i \cdot f(i) - \lambda_1 \cdot \sum_{i=0}^{8} p_i \cdot g(i) - \lambda_2 \cdot \sum_{i=0}^{8} i \cdot p_i$$

Differenting w.r.t. p_j and setting $= 0$

$$\frac{\partial \mathcal{L}}{\partial p_j} = 1 + \ln p_j - \lambda_0 \cdot f(j) - \lambda_1 \cdot g(j) - \lambda_2 \cdot j.$$

$$\Rightarrow p_j = \begin{cases} e^{\lambda_0 + \lambda_2 j - 1} & j \text{ odd} \\ e^{\lambda_1 + \lambda_2 j - 1} & j \text{ even} \end{cases} \qquad \cancel{1} \leq j \leq N$$

which can be rewritten setting $e^{\gamma} = \frac{1}{Z}$

as

$$P_0 = \frac{e^{\lambda_1}}{Z} \qquad P_1 = \frac{e^{\lambda_0+\lambda_2}}{Z} \qquad P_2 = \frac{e^{\lambda_1+2\lambda_2}}{Z}$$

$$P_3 = \frac{e^{\lambda_0+3\lambda_2}}{Z} \qquad P_4 = \frac{e^{\lambda_1+4\lambda_2}}{Z} \qquad P_5 = \frac{e^{\lambda_0+5\lambda_2}}{Z}$$

$$P_6 = \frac{e^{\lambda_1+6\lambda_2}}{Z} \qquad P_7 = \frac{e^{\lambda_0+7\lambda_2}}{Z} \qquad P_8 = \frac{e^{\lambda_1+8\lambda_2}}{Z}$$

(b) the normalization constant Z is determined by

$$\frac{e^{\lambda_1}}{Z} + \frac{e^{\lambda_0+\lambda_2}}{Z} + \frac{e^{\lambda_1+2\lambda_2}}{Z} + \frac{e^{\lambda_0+3\lambda_2}}{Z} + \frac{e^{\lambda_1+4\lambda_2}}{Z} + \frac{e^{\lambda_0+5\lambda_2}}{Z}$$

$$+ \frac{e^{\lambda_1+6\lambda_2}}{Z} + \frac{e^{\lambda_0+7\lambda_2}}{Z} + \frac{e^{\lambda_1+8\lambda_2}}{Z} = 0$$

$$\Rightarrow Z = e^{\lambda_1} + e^{\lambda_0+\lambda_2} + e^{\lambda_1+2\lambda_2} + e^{\lambda_0+3\lambda_2} + e^{\lambda_1+4\lambda_2} + e^{\lambda_0+5\lambda_2}$$

$$+ e^{\lambda_1+6\lambda_2} + e^{\lambda_0+7\lambda_2} + e^{\lambda_1+8\lambda_2}$$

the free energy

$$F = -\ln Z$$

$$= -\ln\left(e^{\lambda_1} + e^{\lambda_0+\lambda_2} + e^{\lambda_1+2\lambda_2} + e^{\lambda_0+3\lambda_2} + e^{\lambda_1+4\lambda_2} + e^{\lambda_0+5\lambda_2} + e^{\lambda_1+6\lambda_2} + e^{\lambda_0+7\lambda_2} + e^{\lambda_1+8\lambda_2}\right)$$

(c)

using the general relation

$$S[\{p_k\}] = -F - \sum_{c=1}^{C} \lambda_c \langle f_c(x) \rangle$$

$$\langle M_o(6) \rangle = -\frac{\partial F}{\partial \lambda_0}$$

$$= \frac{e^{\lambda_0+\lambda_2} + e^{\lambda_0+3\lambda_2} + e^{\lambda_0+5\lambda_2} + e^{\lambda_0+7\lambda_2}}{e^{\lambda_1} + e^{\lambda_0+\lambda_2} + e^{\lambda_1+2\lambda_2} + e^{\lambda_0+3\lambda_2} + e^{\lambda_1+4\lambda_2} + e^{\lambda_0+5\lambda_2} + e^{\lambda_1+6\lambda_2} + e^{\lambda_0+7\lambda_2} + e^{\lambda_1+8\lambda_2}}$$

$$\langle M_e(6) \rangle = -\frac{\partial F}{\partial \lambda_1}$$

$$= \frac{e^{\lambda_1} + e^{\lambda_1+2\lambda_2} + e^{\lambda_1+4\lambda_2} + e^{\lambda_1+6\lambda_2} + e^{\lambda_1+8\lambda_2}}{e^{\lambda_1} + e^{\lambda_0+\lambda_2} + e^{\lambda_1+2\lambda_2} + e^{\lambda_0+3\lambda_2} + e^{\lambda_1+4\lambda_2} + e^{\lambda_0+5\lambda_2} + e^{\lambda_1+6\lambda_2} + e^{\lambda_0+7\lambda_2} + e^{\lambda_1+8\lambda_2}}$$

(d)

setting $\lambda_2 = 0$

$$S[p] = \ln\left(e^{\lambda_1} + e^{\lambda_0+\lambda_2} + e^{\lambda_1+2\lambda_2} + e^{\lambda_0+3\lambda_2} + e^{\lambda_1+4\lambda_2} + e^{\lambda_0+5\lambda_2} + e^{\lambda_1+6\lambda_2} + e^{\lambda_0+7\lambda_2} + e^{\lambda_1+8\lambda_2}\right)$$

$$- \lambda_0 \cdot \frac{e^{\lambda_0+\lambda_2} + e^{\lambda_0+3\lambda_2} + e^{\lambda_0+5\lambda_2} + e^{\lambda_0+7\lambda_2}}{e^{\lambda_1} + e^{\lambda_0+\lambda_2} + e^{\lambda_1+2\lambda_2} + e^{\lambda_0+3\lambda_2} + e^{\lambda_1+4\lambda_2} + e^{\lambda_0+5\lambda_2} + e^{\lambda_1+6\lambda_2} + e^{\lambda_0+7\lambda_2} + e^{\lambda_1+8\lambda_2}}$$

$$- \lambda_1 \cdot \frac{e^{\lambda_1} + e^{\lambda_1+2\lambda_2} + e^{\lambda_1+4\lambda_2} + e^{\lambda_1+6\lambda_2} + e^{\lambda_1+8\lambda_2}}{e^{\lambda_1} + e^{\lambda_0+\lambda_2} + e^{\lambda_1+2\lambda_2} + e^{\lambda_0+3\lambda_2} + e^{\lambda_1+4\lambda_2} + e^{\lambda_0+5\lambda_2} + e^{\lambda_1+6\lambda_2} + e^{\lambda_0+7\lambda_2} + e^{\lambda_1+8\lambda_2}}$$

(e) when $\lambda \to +\infty$

$S[p] \to \ln 8$

when $\lambda \to -\infty$

$S[p] \to \ln 4$

when $\lambda \to 0$

$S[p] \to \ln 9$