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SECOND PUBLIC EXAMINATION

Honour School of Mathematics Part C: Paper C3.10

Master of Science in Mathematical Sciences: Paper C3.10

ADDITIVE AND COMBINATORIAL NUMBER THEORY

TRINITY TERM 2021

Friday 11 June

Opening time: 09:30 (BST)

Mode of completion: Handwritten

You have 1 hour 45 minutes writing time to complete the paper and up to 30 minutes technical time to upload your answer file.

You may submit answers to as many questions as you wish but only the best two will count for the total mark. All questions are worth 25 marks.

You should ensure that you observe the following points:

1. Write with a black or blue pen OR with a stylus on tablet (colour set to black or blue).
2. On the first page, write
   - your candidate number
   - the paper code
   - the paper title
   - and your course title (e.g. FHS Mathematics and Statistics Part C)
   - but ***do not*** enter your name or college.
3. For each question you attempt,
   - start writing on a new sheet of paper,
   - indicate the question number clearly at the top of each sheet of paper,
   - number each page
4. Before scanning and submitting your work,
   - on the first page, in numerical order, write the question numbers attempted,
   - cross out all rough working and any working you do not want to be marked,
   - and orient all scanned pages in the same way.
5. Submit all your answers to this paper as a ***single PDF*** document

If you do not attempt any questions at all on this paper, you should still submit a single page indicating that you have opened the exam but not attempted any questions. Please make sure to write your candidate number on this single page.

1. (a) [3 marks] Let q be a prime, and let a, b be integers, not both congruent to 0 modulo q. Show that

$$\left| \sum_{1 \leqslant n \leqslant q} e\left(\frac{an^2 + bn}{q}\right) \right| \ll q^{1/2}.$$

[*Hint: Complete the square and apply a bound for the Gauss sums.*]

(b) [5 marks] Still letting q be a prime and $a \not\equiv 0 \pmod{q}$, show that

$$\left| \sum_{1 \leqslant n \leqslant q} e\left(\frac{a(n^3 + n)}{q}\right) \right| \ll q^{3/4}.$$

[*Hint: Adapt proofs from the course and use part (a).*]

(c) [7 marks] Show that if q is a large enough prime, then for every $N \in \mathbb{Z}$ there is a solution $(n_1, n_2, \ldots, n_5)$ to the congruence

$$(n_1^3 + n_1) + (n_2^3 + n_2) + \cdots + (n_5^3 + n_5) \equiv N \pmod{q}.$$

[*Hint: Write a formula for the number of solutions in terms of Fourier transforms and use part (b).*]

(d) [10 marks] Show that the conclusion of part (c) continues to hold if the primality condition on q is replaced by the condition $(q, m) = 1$ for some suitable fixed integer $m \geqslant 1$.

[*Hint: You may lift the result of part (c) to prime power moduli by proving the following. If p is odd and a is an integer with $a \not\equiv 0 \pmod{p}$, $27a^2 + 2 \not\equiv 0 \pmod{p}$, then the solvability of $x^3 + x \equiv a \pmod{p}$ implies the solvability of $x^3 + x \equiv a \pmod{p^\ell}$ for every $\ell \geqslant 1$.*]

2. (a) [2 marks] Let $A \subset [N]$. Give a formula for the number of solutions to

$$2x + 3y = 5z, \quad x, y, z \in A \tag{1}$$

involving the Fourier transform of the set A.

(b) [4 marks] Let $A \subset [N]$, $|A| = \delta N$, $\delta > N^{-1/10}$. Show that if all solutions to (1) have $x = y = z$, then

$$\sup_{\alpha \in \mathbb{R}} |\widehat{f_A}(\alpha)| \gg \delta^2 N,$$

where $f_A = 1_A - \delta 1_{[N]}$.

(c) [7 marks] Let $A \subset [N]$ satisfy $|A| = \delta N$ with $N \geqslant N_0(\delta)$ large enough. Sketch a proof that (1) has a solution with x, y, z not all equal.

[*You should give the basic structure of the argument, but need not supply full details.*]

(d) [1 mark] Show that if N is large enough, there exists a set $A \subset [N]$ of size $|A| > N/3$ such that A contains no solutions to

$$2x + 3y = 4z, \quad x, y, z \in A. \tag{2}$$

(e) [11 marks] Consider the set

$$A = \{n \leqslant N : \ \|\sqrt{2}n - 1/3\|_{\mathbb{R}/\mathbb{Z}} \leqslant \frac{1}{100}\}.$$

(i) Show that A contains no solutions to (2).

(ii) Show that if N is large enough in terms of ε, then A is "ε-uniform" in the following sense: for every arithmetic progression $P \subset [N]$ with $|P| \geqslant \varepsilon N$ we have

$$\left| |P \cap A| - \frac{|P||A|}{N} \right| \leqslant \varepsilon |P|.$$

[*You may use the following fact: for any $\delta, \eta > 0$ there exist trigonometric polynomials $T^{\pm}(x) = \sum_{|m| \leqslant M} c_m^{\pm} e^{imx}$ (with $M, c_m^{\pm}$ depending on δ, η) such that*

$$T^-(x) \leqslant 1_{\|x\|_{\mathbb{R}/\mathbb{Z}} \leqslant \delta/2} \leqslant T^+(x)$$

and $c_0^- = \delta(1-\eta)$, $c_0^+ = \delta(1+\eta)$.]

3. (a) [3 marks] Let $N, d \geqslant 1$. Show that the set $[N]^d \subset \mathbb{Z}^d$ is Freiman 100-isomorphic to a subset of $[M] \subset \mathbb{Z}$ for some $M \ll_d N^d$.

(b) [8 marks] Show that there exists $C \geqslant 1$ such that the following holds. Let $A \subset \mathbb{Z}$ satisfy $|A - A| \leqslant 100|A|$ with $|A|$ large enough. Then $|kA| \leqslant k^C |A|$ for every integer $k \geqslant 1$.

[*You may assume the following fact: in the Freiman–Ruzsa theorem, the generalised arithmetic progression can be taken to be proper.*]

(c) [7 marks] Let $K \geqslant 1$ be large enough. Show that there exists $A \subset \mathbb{Z}$ satisfying $|A + A| \leqslant K|A|$ and such that A is not contained in any proper generalised arithmetic progression of dimension $\leqslant K - 1$ and size $\leqslant 2^{o(K)}$.

[*Hint: Take A to be a suitable geometric progression.*]

(d) [7 marks] Show that there exists $c > 0$ such that the following holds. There exist arbitrarily large sets $A \subset \mathbb{Z}$ with

$$|A - A| > |A + 2.A|^{1+c},$$

where we denote $\lambda.A = \{\lambda a : \ a \in A\}$.