1)

$\tau(x,y) = (x+1, y+2)$, $P=(1,1)$

$\exists Q \in \mathbb{R}^2$, $\tau = \sigma_Q \sigma_P$

$R = \tau(P) = (1+1, 1+2) = (2,3)$

$Q = \frac{P+R}{2} = (\frac{3}{2}, 2)$

We show $\tau = \sigma_Q \sigma_P$

We know for $M=(a,b)$, $\sigma_M(x,y) = (-x+2a, -y+2b)$

$\sigma_Q \sigma_P(x,y) = \sigma_Q(-x+2(1), -y+2(1)) =$

$(-(-x+2)+2(\frac{3}{2}), -(-y+2)+2(2)) = (x+1, y+2) = \tau(x,y)$

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2)

$\alpha(x,y) = (x+y, xy)$

a) No. α is not one-to-one.

We prove $\exists$ $P=(a,b)$ and $Q=(c,d)$ such that $\alpha(P) = \alpha(Q)$ but $P \neq Q$.

Take $P=(2,1)$ and $Q=(1,2)$ where

$\alpha(P) = (3,2) = \alpha(Q)$ but $P \neq Q$.

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b) $l: x-y=0$ Is $\alpha(l)$ a line? No.

$(1,1)$ and $(2,2)$ and $(3,3) \in l$.

$\alpha(1,1) = (2,1)$
$\alpha(2,2) = (4,4)$
$\alpha(3,3) = (6,9)$
} non collinear

Actually: $\begin{cases} x' = x+y \\ y' = xy \end{cases}$ if $x=y \rightarrow \begin{cases} x' = 2x \rightarrow x = \frac{x'}{2} \\ y' = x^2 \end{cases}$

$\Rightarrow y' = x^2 = (\frac{x'}{2})^2 = \frac{1}{4}x'^2$

the image of l is the parabola $y = \frac{1}{4}x^2$.

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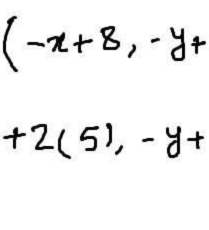
3) $A=(1,1)$ $B=(3,2)$ $C=(4,4)$

a) $\tau_{AB}\sigma_C = \sigma_P$ for some P

$\tau_{AB} = (2,1)$

$\tau_{AB} = \sigma_P \sigma_C$



$P = \frac{C+\tau(C)}{2} = (5, \frac{9}{2})$

$\tau_{AB}\sigma_C(x,y) = \tau_{AB}(-x+8, -y+8) = (-x+10, -y+9)$

$\sigma_P(x,y) = (-x+2(5), -y+2(\frac{9}{2})) = \tau_{AB}\sigma_C(x,y)$

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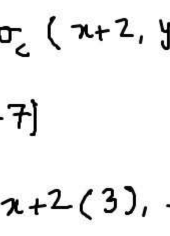
b) $\sigma_C \tau_{AB} = \sigma_Q$ for some Q

$\tau_{AB} = \sigma_C \sigma_Q$



$C = \frac{Q+\tau(Q)}{2} \rightarrow \begin{cases} 4 = a+1 \rightarrow a = 3 \\ 4 = b + \frac{1}{2} \rightarrow b = \frac{7}{2} \end{cases} \Rightarrow Q = (3, \frac{7}{2})$

$\sigma_C \tau_{AB}(x,y) = \sigma_C(x+2, y+1) = (-(x+2)+8, -(y+1)+8) =$

$(-x+6, -y+7)$

$\sigma_Q(x,y) = (-x+2(3), -y+2(\frac{7}{2})) = \sigma_C \tau_{AB}(x,y)$

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c) $\sigma_C \tau_{AB} \sigma_C = \tau_{\sigma_C(A)\sigma_C(B)}$

$\sigma_C \tau_{AB} \sigma_C(x,y) = \sigma_C \tau_{AB}(-x+8, -y+8) = \sigma_C(-x+10, -y+9) =$

$(-(-x+10)+2(4), -(-y+9)+2(4)) = \boxed{(x-2, y-1)}$

$\sigma_C(A) = \sigma_C(1,1) = (-1+8, -1+8) = (7,7)$

$\sigma_C(B) = \sigma_C(3,2) = (-3+8, -2+8) = (5,6)$

$\tau_{\sigma_C(A)\sigma_C(B)}(x,y) = (x,y) + \sigma_C(B) - \sigma_C(A) = (x,y) + (5-7, 6-7) =$

$\boxed{(x-2, y-1)}$

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d) $\sigma_B \rho_{A,\frac{\pi}{2}} \sigma_B = \rho_{\sigma_B(A), \frac{\pi}{2}}$

$\rho_{A,\frac{\pi}{2}}(x,y) = (-y+2, x) \rightarrow$ (From Assignment 2)

$\sigma_B \rho_{A,\frac{\pi}{2}} \sigma_B(x,y) = \sigma_B \rho_{A,\frac{\pi}{2}}(-x+6, -y+4) =$

$\sigma_B(-(-y+4)+2, -x+6) = \sigma_B(y-2, -x+6) =$

$(-(y-2)+6, -(-x+6)+4) = \boxed{(-y+8, x-2)}$

$\sigma_B(A) = \sigma_B(1,1) = (-1+6, -1+4) = (5,3)$

$\rho_{\sigma_B(A),\frac{\pi}{2}}(x,y) = \boxed{(-y+8, x-2)}$

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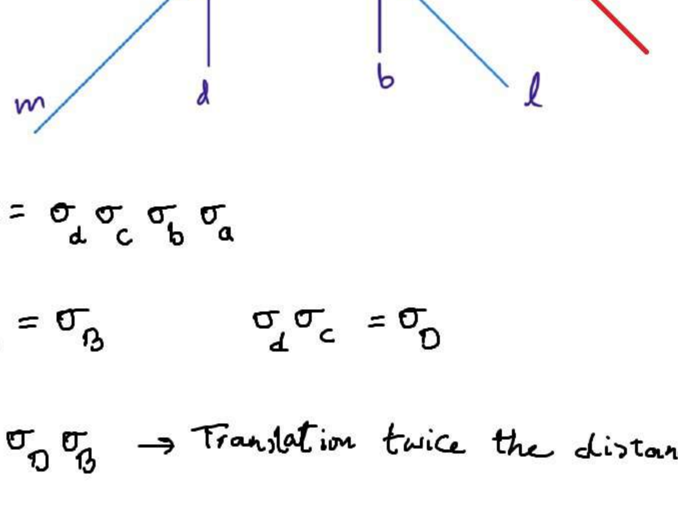
4)



a) $\alpha = \sigma_d \sigma_c \sigma_b \sigma_a$

$\sigma_b \sigma_a = \sigma_B$ $\sigma_d \sigma_c = \sigma_D$

$\rightarrow \alpha = \sigma_D \sigma_B \rightarrow$ Translation twice the distance from B to D

b) $\alpha = \sigma_c \sigma_d \sigma_a \sigma_b$

$\sigma_a \sigma_b = \sigma_b \sigma_a = \sigma_B$ $\sigma_c \sigma_d = \sigma_d \sigma_c = \sigma_D$

$\rightarrow \alpha = \sigma_D \sigma_B \rightarrow \ldots$

c) $\alpha = \sigma_a \sigma_b \sigma_c \sigma_a = \sigma_a \sigma_C \sigma_a = \sigma_{\sigma_a(C)}$

$\rightarrow$ halfturn centered at the reflection of point C in line a.

d) $\beta = \sigma_l \sigma_m \sigma_l$

$m \perp l \Rightarrow \sigma_m \sigma_l = \sigma_l \sigma_m \rightarrow \beta = \sigma_l \sigma_l \sigma_m = \sigma_m$

e) $\beta = \sigma_b \sigma_m \sigma_a$

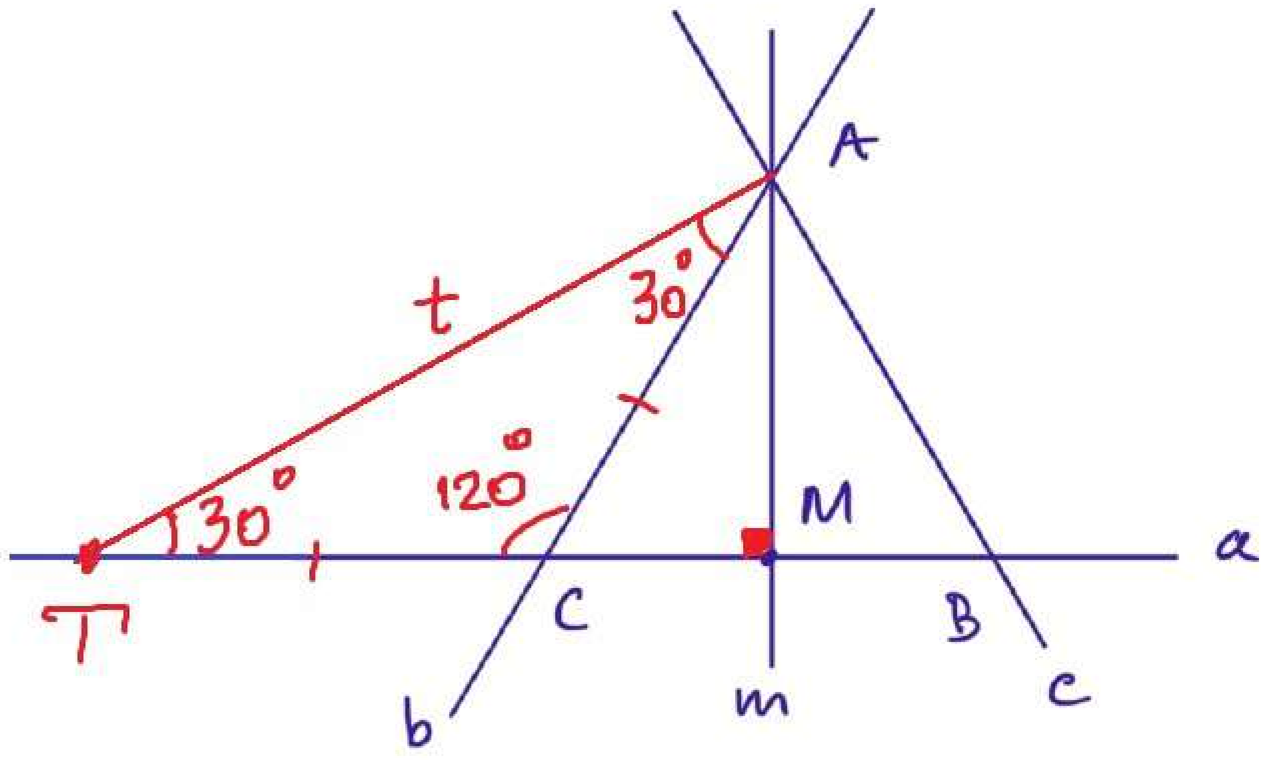
$\sigma_m \sigma_a = \sigma_b \sigma_m = \rho_{B,\frac{\pi}{2}}$

$\rightarrow \beta = \sigma_b \sigma_b \sigma_m = \sigma_m$

f) $\beta = \sigma_a \sigma_m \sigma_a = \sigma_{\sigma_a(m)}$

$\rightarrow$ reflection in the image of line m in line a, i.e., line $t \parallel l$ passing thru B, as shown above)

5)



a)

$$\alpha = \rho_{B,120°}\ \rho_{A,120°}$$

$\rho_{A,120°} = \sigma_c \sigma_b$ $\qquad$ $\rho_{B,120°} = \sigma_a \sigma_c$

$$\rightarrow \alpha = \sigma_a \sigma_c \sigma_c \sigma_b = \sigma_a \sigma_b = \rho_{C,120°}$$

b)

$$\alpha = \rho_{B,120°}\ \rho_{A,60°}$$

$\rho_{B,120°} = \sigma_a \sigma_c$ $\qquad$ $\rho_{A,60°} = \sigma_C \sigma_m$

$$\rightarrow \alpha = \sigma_a \sigma_c \sigma_c \sigma_m = \sigma_a \sigma_m = \sigma_M$$

c)

$$\alpha = \sigma_c \sigma_m \sigma_b \sigma_a$$

$$\sigma_c \sigma_m = \sigma_m \sigma_b \quad \rightarrow \alpha = \sigma_m \sigma_b \sigma_b \sigma_a = \sigma_m \sigma_a = \sigma_M$$

d)

$$\alpha = \sigma_m \sigma_c \sigma_b \sigma_a$$

$$\sigma_m \sigma_c = \sigma_t \sigma_b$$

$$\rightarrow \alpha = \sigma_t \sigma_b \sigma_b \sigma_a = \sigma_t \sigma_a = \rho_{T,30°}$$