Question 1.

proof:

(a)

$\phi \in \tau$. $\{1.2.3.4\} \notin \tau$. so (O1) is not satiesfied by τ.

$\{1.3\}$. $\{2.3.4\} \in \tau$. and $\{3\} = \{1.3\} \cap \{2.3.4\} \notin \tau$.

so (O2) is not satiesfied by τ.

$\{1\} \in \tau$. $\{2.3.4\} \in \tau$ and $\{1.2.3.4\} = \{1\} \cup \{2.3.4\} \notin \tau$.

so (O3) is not satiesfied by τ.

So τ does not satiesfied (O1). (O2) (O3) and it is not a topology on X

(b) $\phi \in \tau$. $X \in \tau$. so (O1) is satiesfied by τ.

$\{3,4\} \cap \{1.2.3.4\} = \{3.4\} \in \tau$. $\{1.2.3.4\} \cap \{3.4.5.6\} = \{3.4\} \in \tau$.

$\{3.4\} \cap \{3.4.5.6\} = \{3.4\} \in \tau$.

so τ satiesfied (O2).

and it is not hard to check $\forall A_1. A_2 \in \tau$.

$A_1 \cup A_2 \in \tau$. so because τ is a finite set.

$\forall$ I arbitrary index set. if $G_i \in \tau$. $\forall i \in I$.

then we know $\bigcup_{i \in I} G_i \in \tau$. so τ satiesfied (O3)

so τ satiesfied (O1), (O2). (O3) and it is a topology on X.

(c)

$X=\mathbb{R}$. $\tau=\{\mathbb{R}, A\subseteq\mathbb{R} \mid \mathbb{R}\setminus A$ is an uncontable set$\}$

then because $\emptyset=\mathbb{R}\setminus\mathbb{R}$. $\mathbb{R}$ is an uncontable set.

so $\emptyset\in\tau$ and by definition $\mathbb{R}\in\tau$.

so $\emptyset, \mathbb{R}\in\tau$. τ satiesfied (O1).

now if $n\in\mathbb{N}^*$. $O_i=\mathbb{R}\setminus\widetilde{O_i}\in\tau$. $\forall\ i\in\{1,2,\cdots,n\}$

then $\bigcap_{i=1}^{n} O_i = \bigcap_{i=1}^{n}(\mathbb{R}\setminus\widetilde{O_i}) = \mathbb{R}\setminus(\bigcup_{i=1}^{n}\widetilde{O_i})$

~~so~~ and because O_1 is a uncontable set.

$O_1 \subseteq (\bigcup_{i=1}^{n}\widetilde{O_i})$ so $\bigcup_{i=1}^{n}\widetilde{O_i}$ is a uncountable set.

so $\mathbb{R}\setminus(\bigcup_{i=1}^{n}\widetilde{O_i})\in\tau$. i.e. $\bigcap_{i=1}^{n} O_i\in\tau$.

so τ satiesfied (O2)

now. $\mathbb{R}=\{x \mid x\in\mathbb{R}\}$. we know $\forall x\in\mathbb{R}$ $\{x\}=\mathbb{R}\setminus(\mathbb{R}\setminus\{x\})$

because $\mathbb{R}\setminus\{x\}$ is a uncontable set.

so $\{x\}\in\tau$. $\forall x\in\mathbb{R}$

but $\mathbb{R}\setminus\{0\}=\bigcup_{\substack{x\neq 0\\ x\in\mathbb{R}}}\{x\}$. if (O3) hold then

$\mathbb{R}\setminus\{0\}\in\tau$. but $\{0\}$ is a contable (in fact finite) set.

this is a contradiction with the definition of τ.

so (O3) is not hold for τ.

so τ is not a topology of X. $\blacksquare$

Question 2.

proof:

(a) because $\mathcal{B}(x)$ is a neighbourhood basis of x in topology τ on X

so $\forall$ neighbourhood $V \in \mathcal{U}(x)$ $\exists B \in \mathcal{B}(x)$. s.t $B \subseteq V$.

now. to prove $\{B \cap A \mid B \in \mathcal{B}(x)\}$ is a neighbourhood basis of x in the induced topology $\tau|_A$ on A.

we first prove: $\forall Y \in \{B \cap A \mid B \in \mathcal{B}(x)\}$. Y satisfied there exist an open set G. in topology $\tau|_A$. s.t $x \in G \subseteq Y$.

in fact $\exists B \in \mathcal{B}(x)$. s.t $Y = B \cap A$. and $\exists G$ open in τ. s.t $x \in G \subseteq B$. so $x \in G \cap A \subseteq B \cap A = Y$.

and because A is open in topology $\tau|_A$. so $\forall Y \in \{B \cap A \mid B \in \mathcal{B}(x)\}$ $\exists$ open set G. s.t. $x \in G \subseteq Y$.

and on the other hand, we prove: $\forall V \in \mathcal{U}(x)$ is a neighbourhood of x in $\tau|_A$. $\exists Y \in \{B \cap A \mid B \in \mathcal{B}(x)\}$. s.t.

$x \in B \subseteq V$.

in fact. $\forall V \in \mathcal{U}(x)$ for $(A, \tau|_A)$ we know $\exists \tilde{V} \in \mathcal{U}(x)$ for (X, τ)

s.t $V = A \cap \tilde{V}$. and for $\tilde{V}$ by the definition of $\mathcal{B}(x)$

we know there exist $B \in \mathcal{B}(x)$. s.t $x \in B \subseteq \tilde{V}$.

so $x \in B \cap A \subseteq \tilde{V} \cap A = V$

so $\forall V \in \mathcal{U}(x)$. for $(A, \tau|_A)$. we can find $Y \in \{B \cap A \mid B \in \mathcal{B}(x)\}$

s.t. $x \in Y \subseteq V$.

so $\{B \cap A \mid B \in \mathcal{B}(x)\}$ is a neighbourhood basis of x. in the induced topology $\tau|_A$ on A.

(b) when $0 < x \leq 1$.

$\{B \cap A \mid B \in \{(y.x] \mid y \in \mathbb{R}, y < x\}\}$ is a neighbourhood basis of x in $\tau|_A$ on A by (a)

and $\forall y \in \mathbb{R}$. $(y.x] \cap [0.1] = \begin{cases} [0.x] & y < 0 \\ (y.x] & 0 \leq y \end{cases}$.

so when $0 < x \leq 1$.

a neighbourhood basis of x in the induced topology $\tau|_A$

is $\{(y.x) \mid 0 \leq y < x\} \cup \{[0.x]\}$.

when $x = 0$ $\mathcal{B}(x) = \{(y.0] \mid y \in \mathbb{R} \ \ y < 0\}$.

so $\forall B \in \mathcal{B}(x)$. $B \cap A = (y.0] \cap [0.1] = \{0\}$.

so when $x = 0$.

a neighbourhood basis of x in the induced topology $\tau|_A$ is $\{\{0\}\}$ $\blacksquare$

Question 3.

proof: consider $(\mathbb{R}, \sigma)$ and σ is the standard metric topology on $\mathbb{R}$

so any open set $O \in \sigma$ satisfied. $\forall x \in O$ $\exists r \in \mathbb{R}^+$

s.t. $x \in B_r(x) \subseteq O$

to prove $(\mathbb{R}, \sigma)$ is second countable we only need to

prove there is a countable subset $\mathcal{B} \subseteq \tau$. s.t. $\forall$

$O \in \tau$. O is a countable union of element in $\mathcal{B}$.

we claim $\mathcal{B} = \{ B_r(x) \mid r \in \mathbb{Q}^+ . \ x \in \mathbb{Q} \}$ satisfied the condition

in fact.

for an arbitrary open set $O \subseteq \mathbb{R}$. $\forall x \in O$. there exist. $r > 0$.

$B_r(x) \subseteq O$. now ~~because~~ because $\mathbb{Q}$ is dense in $\mathbb{R}$, so there

exist $\tilde{x} \in B_r(x) \cap \mathbb{Q}$. and we can find a radius $\tilde{r} \in \mathbb{Q}^+$.

s.t. $d(x, \tilde{x}) < \tilde{r} < r$ (this can be done because $\mathbb{Q}^+$ is dense in $\mathbb{R}^+$)

now we know $O = \{ x \mid x \in O \} \subseteq \bigcup_{x \in O} B_{\tilde{x}}(\tilde{r})$

where $\tilde{x}, \tilde{r} \in \mathbb{Q}$. $\tilde{r} > 0$ but $\mathcal{B} = \{ B_{\tilde{x}}(\tilde{r}) \mid \tilde{x} \in \mathbb{Q}. \ \tilde{r} \in \mathbb{Q}^+ \}$ itself.

is a countable set.

so $O \subseteq \bigcup_{x \in O} B_{\tilde{x}}(\tilde{r})$ is a union of countable ~~sets~~ element in $\mathcal{B}$

this is true for all $U \in G$ is an open set.

So G is a second countable topology in $\mathbb{R}$ $\blacksquare$

Question 4.

proof: for any set $A \in X$. X equipped topology τ

we know $\mathring{A}$ is the set of interior point of A

So to prove $(A \cup B)^\circ \supseteq A^\circ \cup B^\circ$

we need to prove $\forall x \in A^\circ \cup B^\circ$, $x \in (A \cup B)^\circ$

$x \in A^\circ \cup B^\circ \Rightarrow A$ is a neighborhood of x or B is a neighborhood of x

$\Rightarrow \exists O$ open set. $x \in O \subseteq A$ or $x \in O \subseteq B$.

in any case $\Rightarrow \exists O$ open set. $x \in O \subseteq A \cup B$

$\Rightarrow x \in (A \cup B)^\circ$.

So $(A \cup B)^\circ \supseteq \mathring{A} \cup B^\circ$

on the other hand. in general $(A \cup B)^\circ \subseteq A^\circ \cup B^\circ$ does not hold

an counterexample is $X = \{1, 2, 3, 4\}$ $G = \{\emptyset, \{1, 2, 3\}, X\}$.

it is not hard to check G is a topology of X.

now take $A = \{1, 2\}$. $B = \{1, 3\}$ and $x = 1$ $A \cup B = \{1, 2, 3\}$.

then by definition of interior point. $x=1 \in \{1,2,3\} \subseteq A\cup B$.

and $\{1,2,3\}$ is open. so $x \in (A\cup B)^{\circ}$.

but. $x \notin \mathring{A}$ or $\mathring{B}$

so in general $(A\cup B)^{\circ} \supseteq \mathring{A} \cup \mathring{B}$ does not hold #

Question 5.

proof: to prove $f(x,y)=x+y$ $\forall (x,y)\in \mathbb{R}^2$ is continuous

we only need to prove $\forall$ open set $O \in \mathbb{R}$. $f^{-1}(O)$ is open in $\mathbb{R}^2$. because both $\mathbb{R}, \mathbb{R}^2$ are equipped standard metric topolgy

so suffice to prove: $\forall (x,y) \in f^{-1}(O)$. $\exists r>0$ s.t. $B_r((x,y)) \subseteq f^{-1}(O)$

because $f(x,y) \in O$ so $\exists \tilde{r}>0$. s.t $\left(f(x,y)-\tilde{r}, f(x,y)+\tilde{r}\right) \subseteq O$

so we suffice to prove : $\exists r>0$. s.t.

$$\forall (\tilde{x},\tilde{y}) \in B_r((x,y)) \quad f(x,y)-\tilde{r} < f(\tilde{x},\tilde{y}) < f(x,y)+\tilde{r} \quad \cdots (*)$$

i.e. $\exists r>0$. s.t. $\forall$ $\sqrt{(x-\tilde{x})^2+(y-\tilde{y})^2}<r$.

$x+y-\tilde{r} < \tilde{x}+\tilde{y} < x+y+\tilde{r}$.

in fact. when we take $r=\frac{\tilde{r}}{3}$

$|x-\tilde{x}| \leqslant \sqrt{(x-\tilde{x})^2+(y-\tilde{y})^2} < r \leqslant \frac{\tilde{r}}{3}$

$|y-\tilde{y}| \leqslant \sqrt{(x-\tilde{x})^2+(y-\tilde{y})^2} < r \leqslant \frac{\tilde{r}}{3}$

so $\quad \tilde{x}+\tilde{y} \leq x+y+|x-\tilde{x}|+|y-\tilde{y}| \leq x+y+\frac{2}{3}\tilde{r} < x+y+\tilde{r}$.

$x+y-\tilde{r} \leq x+y-\frac{2}{3}\tilde{r} < x+y-|x-\tilde{x}|-|y-\tilde{y}| \leq \tilde{x}+\tilde{y}$

so when we ~~take~~ $r=\frac{\tilde{r}}{3}$

we know $\forall (\tilde{x},\tilde{y}) \in B_r((x,y))$.

$$\left| f(x,y)-f(\tilde{x},\tilde{y}) \right| < \tilde{r}$$

so (*) is ~~true~~ and f is a continuous map.

f is not a homeomorphism. in fact if f is a homeomorphism then f is a bijective.

but $f(0,1) = f(1,0) = 1$ so f is not an injective. So it is not a bijective.

So f is not a homeomorphism

$\boxtimes$